If one vector is a multiple of another, then the vectors point in the same direction but have different lengths. (1, 2) and (5, 10) point in the same direction because (5, 10) = 5(1, 2) but (5, 10) will be 5 times as long as (1, 2). If a vector is a negative multiple of another, then it will point in the opposite direction. Since (-5, -10) = -5(1, 2), that means that (-5, -10) will be 5 times as long as (1, 2) and will point in the opposite direction.

Dot Product and Cross Product

To find the dot product of two vectors, simply multiply the respective coordinates together and then add the results. For example, (1, 2, 1) • (2, 0, -3) = 1X2 + 2X0 + 1X(-3) = 2 + 0 + (-3) = -1

To find the cross product of two vectors:

	Example: (1, 3, 2) x (4, 1, 0)
Step 1: Write the coordinates of the first vector horizontally twice	1 3 2 1 3 2
Step 2: Do the same to the second vector. Put this row directly below the first	1 3 2 1 3 2 4 1 0 4 1 0
Step 3: Eliminate the first and last columns	3 2 1 3 1 0 4 1
Step 4: Make an "X" between the first two columns. The first coordinate will be: 3X0 – 2X1 = 0 – 2 = -2	3 2 1 3 1 0 4 1 (1, 3, 2) x (4, 1, 0) = (-2, ?, ?)
Step 5: Move the "X" down a step and do the same thing. The second coordinate is: 2X4 – 0X1 = 8 – 0 = 8	3 2 1 3 1 0 4 1 (1, 3, 2) x (4, 1, 0) = (-2, 8, ?)
Step 6: Move the "X" down one last step and repeat. The third coordinate is: 1X1 – 3X4 = 1 – 12 = -11	3 2 1 3 1 0 4 1 (1, 3, 2) x (4, 1, 0) = (-2, 8, -11)

(This might sound complicated, but it's one of those things that seems harder when put into words. If you try it out, it's not that hard. And it's a lot easier than the method taught in the textbook or in class.)

Length and Unit Vectors

To find the length of a vector, square each coordinate and add them together. Then, take the square root of the result. The length is denoted by double vertical bars.

To find the length of (1, 4, -2):

$$\|(1,4,-2)\|$$
$$=\sqrt{1^2+4^2+(-2)^2}$$
$$=\sqrt{1+16+4}$$
$$=\sqrt{21}$$

(Length can also be called norm or magnitude)

A unit vector is a vector with length one. If you had vector and you wanted to find a unit vector pointing in the same direction, start by finding the length of that vector and then divide each coordinate by that length. (If you had a 10 foot pole and you wanted to scale it down to 1 foot, you'd divide it by 10. Likewise, if you had a $\sqrt{3}$ foot pole and you wanted to scale it down to 1 foot, you'd divide it by $\sqrt{3}$)

So, a unit vector pointing in the direction of (1, 4, -2) is:

$$\frac{1}{\sqrt{21}}(1,4,-2)$$

Parallel and Perpendicular Vectors

Two vectors are parallel if each of their coordinates are proportional – in other words, one vector should be a multiple of the other. For example, (1, 2, 1) and (4, 8, 4) are parallel because (4, 8, 4) = 4(1, 2, 1)

Two vectors are perpendicular if their dot product is equal to zero.

The dot product is used to test if two vectors are perpendicular. It always returns a number.
The dot product is useful for questions like “Find the value of k for which ... and ... are perpendicular”

The cross product is used to generate a new vector perpendicular to two given vectors. It always returns a vector. The cross product is useful for questions like “Find a vector perpendicular to ... and ...”

The dot product is also useful in questions that ask “Find the cosine of the angle between u and v” by applying the following formula:

$$\cos\theta=\frac{u\cdot v}{\|u\|\|v\|}$$

Area and Volume

The area of a triangle is

$$A = \frac{1}{2} \| u \times v \|$$

The area of a parallelogram is

$$A = \| u \times v \|$$

The volume of a parallelopiped (it's a three dimensional parallelogram – like a cube that got bent on its side) is

$$V = | u \cdot (v \times w) |$$

In the formulas above, u, v, w are the vectors that determine the parallelogram or parallelopiped. They'll be given to you in the question.

Equations of Lines

1. Point Parallel Form (or Point Vector Form)

To use this form, you need **a point (p)** on the line and **a vector (v)** pointing in the direction of the line (called a direction vector).

(x, y, z) = p + tv

So, if we wanted a line in the direction (1, 2, 1) going through (4, -1, 0), the point parallel form would be: (x, y, z) = (4, -1, 0) + t(1, 2, 1)

t is called a parameter. By plugging in different values for t, the equation will spit out different points on the line.

2. Point Normal Form

To use this form, you need a point (p) on the line and a vector perpendicular to the line. This vector is called a normal (n).

((x, y) – p) * n = 0

Point Normal Form gives a line if we're in R2. In higher dimensions, point normal form is used for planes.

3. Two Point Form

To use this form, you need two points on the line (p and q). The equation is:

(x, y, z) = tp + (1-t)q

For each of these equations, you can choose any point on the line or any direction vector. So, the

61. Find the volume of the parallelepiped with edges given by the vectors $u=(1,0,1)$, $v=(2,1,-3)$ and $w=(2,-1,2)$

A: 1	B: 2	C: 3	D: 4	E: 5

62. Find $\cos\theta$ where θ is the angle between the vectors u = (1, 1, 1, 1) and w = (1, 2, 0, 2)

A: 1	B: 5/6	C: 1/6	D: 5/36	E: 1/36

63. If u = (2, 1, 1) and v = (4, k, 2) are orthogonal (perpendicular), find k

A: 2	B: -6	C: 10	D: 6	E: -10

64. If $u=(1,-1,3)$ and $v=(2,1,-1)$, then $u\times v=$

A: (2, -1, -3)	B: (-2, 7, 3)	C: (-2, -7, 3)	D: (-3, 7, 3)	E: (-3, 5, 4)

65. Find the volume of the parallelepiped with edges given by the vectors u = (1, 1, 1), v = (0, 2, 1) and w = (1, 1, 5)

66. Consider the vectors $u=(1,0,1)$, $v=(0,1,1)$ and $w=(1,1,1)$
Then:

$$\|u\|=\sqrt{2} \qquad u\cdot u=2 \qquad u\times u=(0,0,0)$$
$$\|v\|=\sqrt{2} \qquad u\cdot v=1 \qquad u\times v=(-1,-1,1)$$
$$\|w\|=\sqrt{3} \qquad u\cdot w=2 \qquad u\times w=(-1,0,1)$$

Use these facts to solve the following three questions:

i. Find $\cos\theta$, where θ is the angle between the vectors u and v

A: $\frac{1}{2}$	B: $\frac{1}{\sqrt{2}}$	C: -1	D: 1	E: $\sqrt{3}$

ii. Find the area of the parallelogram determined by the vectors u and v

A: $\frac{1}{2}$	B: $\frac{1}{\sqrt{2}}$	C: -1	D: 1	E: $\sqrt{3}$

iii. Find the volume of the parallelepiped determined by the vectors u, v and w.

A: $\frac{1}{2}$	B: $\frac{1}{\sqrt{2}}$	C: -1	D: 1	E: $\sqrt{3}$

67. Find the unit vector with the same direction as (1, 2).

68. Given $u=(1,2,3)$, $v=(-3,-2,-1)$ and $w=(-1,1,2)$, and that $u\times v=(4,-8,4)$, find the volume of the parallelepiped determined by the vectors u, v and w.

A: -4	B: 4	C: $\sqrt{96}$	D: 2	E: $\sqrt{6}$

Section 1.3 – 2.1: Lines and Planes

- Equations of Lines
- Equations of Planes
- Intersection of two lines
- Intersections of a line and a plane
- Normal vectors
- Direction vectors
- Parallel and perpendicular lines

69. Find an equation in point-parallel form of the line passing through the points P(0, 1, -2, 3) and Q(3, 2, -1, 0)

70. Find the point of intersection of the following two lines:

$$l_1 \quad : \quad (x, y) = (4, -1) + s(1, -1)$$
$$l_2 \quad : \quad (x, y) = (5, 5) + t(3, 4)$$

71. An equation in point-normal form for the plane π is $(3,1,2)\cdot\left(x-(1,2,1)\right)=0$. What is an equation for this plane in standard form?

A: $x+2y+z=7$
B: $x+2y+z=0$
C: $3x+y+2z=0$
D: $3x+y+2z=7$
E: None of A, B, C, D

72. Let π be the plane through Q(1, 0, 0) with normal n = (3, 0, 4). Find the distance between this plane and the point P(1, 1, 1)

A: 4/5	B: 3/5	C: 2/5	D: 1/5	E: 0